**Scientific Report**

*Concerning the implementation of the project in the period May 2013 - December 2014*

**Title of the project:** *Surfaces making constant angle with a Killing vector field in $M^2 \times R$*

**Code:** PNII-RU-PD-2012-3-0387, contract no.37/26.04.2013

**Project leader:** Asist.univ.dr. Ana Irina Nistor

**Mentor:** Conf.univ.dr. Dorel Fetcu

**Main objective:** the study of constant angle property for curves and surfaces in ambient product spaces.

**Description of the research activity:**

The results we have obtained so far are included in the following articles (published, accepted, or only under review):

1. A.I. Nistor, „Surfaces making constant angle with a Killing vector field in $S^2 \times R$‟, submitted.

Let us recall first the definition of a constant angle surface, namely an oriented surface for which the unit normal in each point makes constant angle with a fixed direction from the ambient space. When the ambient space is a 2-parameter solvable Lie group and the fixed direction is chosen as one of the left invariant vector fields, the classification results were finalized in the first stage of this project, May-December 2013, and the corresponding article [4] was published in November 2014.

In order to complete the first proposed objective, that is the study of curves and surfaces which make constant angle with a Killing vector field in the product space $S^2 \times R$, where $S^2$ represents the unit sphere, the research activity consisted in developing another direction of study, represented by the generalization of the results obtained in the classification of constant angle surfaces in product spaces $M^2(c) \times R$ when the fixed direction was the real axis $R$. Here $M^2(c)$ denotes a surface of constant Gaussian curvature $c$, including the plane $R^2$ for $c=0$, the unit sphere $S^2$ for $c=1$ and the hyperbolic plane $H^2$ for $c=-1$. For the beginning we turned our attention to the ambient space $S^2 \times R$. We know that the isometry group of $S^2 \times R$ has dimension 4, thus we have 4 fundamental Killing vector fields: a translational Killing vector field along $R$, $d/dt=(0,0,0,1)$, where $t$ denotes the global coordinate on $R$, and 3 rotational Killing vector fields $(-y,x,0,0), (z,0,-x,0),(0, z,-y,0)$, where $(x,y,z)$ represent the coordinates of a point in the sphere $S^2$ isometrically immersed in $R^3$. 

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First we classified curves which make constant angle with the translational Killing vector field $d/dt$ and the rotational one $V= (-y, x, 0, 0)$:

**Theorem.** A curve in $S^2 \times \mathbb{R}(t)$ makes constant angle with the Killing vector field $d/dt$ if and only if it is given, up to isometries of the ambient space, by the geodesics of $S^2 \times \mathbb{R}$.

**Theorem.** A curve $\gamma$ in $S^2 \times \mathbb{R}(t)$ makes constant angle with the Killing vector field $(-y, x, 0, 0)$ if and only if it is given in the $(\phi, \psi, t)$ coordinates, up to isometries of the ambient space, as follows:

$$\phi(s) = \sin \theta \int \cos \omega(\zeta) d\zeta, \quad \psi(s) = \cos \theta \int \frac{d\zeta}{\cos \phi(\zeta)}, \quad t(s) = \sin \theta \int \sin \omega(\zeta) d\zeta,$$

where $\omega$ is a smooth function defined on a real interval, and $(\phi, \psi)$ represent the spherical coordinates.

In the case of surfaces, we used spherical coordinates on $S^2$ and we found the explicit parametrizations of surfaces which make constant angle with $V$, using the almost contact metric structure of $S^2 \times \mathbb{R}$. As we anticipated, we succeeded to obtain the equations of the immersion and to solve them with the help of elliptic functions, obtaining the following classification result:

**Theorem.** A surface in $S^2 \times \mathbb{R}(t)$ makes constant angle with the rotational Killing vector field $V= (-y, x, 0, 0)$ if and only if it is parametrized by:

$$F(u, v) = (\cos \phi(u, v) \cos \psi(u, v), \cos \phi(u, v) \sin \psi(u, v), \sin \phi(u, v), t(u, v)),$$

where the coordinate functions are given by:

$$\phi(u, v) = \cos \theta \int \cos \alpha(\zeta) d\zeta, \quad \psi(u, v) = \frac{\tan \theta}{\sqrt{2}} v + \frac{\sin \theta}{\sqrt{2}} \int \frac{d\zeta}{\sin \alpha(\zeta)}, \quad t(u, v) = v + \cos \theta \int \sin \alpha(\zeta) d\zeta,$$

and $\alpha(u) = am\left(\frac{1}{\sqrt{m}} \cos \theta + c \mid m\right), m, c \in \mathbb{R}$, where "am" denotes the Jacobi amplitude.

The article [1] containing these results was submitted for publication.

Further-on, we will approach the problem in the product space $H^2 \times \mathbb{R}$, anticipating to obtain classification results for curves and surfaces which make constant angle with a Killing vector field. Discussions in this direction took place with prof. Palmas and prof. Vrancken, who visited the Department of Mathematics and Informatics of the Technical University „Gh. Asachi” from Iasi in the period May-June 2014.

Next, we continued the study of magnetic curves, study previously started by the project leader. The motivation to continue to develop this working direction is given by, for example, the fact that in the Euclidean 3-space the magnetic curves are helices (with the particular cases of circles and lines) and it is known that the helices make constant angle with the Killing vector field corresponding to one of the coordinate axes on $\mathbb{R}^3$.

The constant angle property is found as a consequence in the study of magnetic curves. A magnetic curve describes the trajectory of a particle which evolves in a magnetic field, being the solution of Lorentz equation. On a Riemannian manifold, a magnetic field is defined by a closed 2-form. Moreover, in the absence of the magnetic field, the particle moves freely under the action of the gravitation, its
trajectory being described by a geodesic in the ambient space. Thus, we may consider magnetic curves as a generalization of geodesics.

At the conference 9-th Brazilian Colloquium of Mathematics, IMPA, Rio de Janeiro, Brazil, in the period 21.07.2013-02.08.2013, the project leader presented a poster entitled „Complete classification of magnetic trajectories in cosymplectic manifolds“ containing the results obtained in the study of magnetic curves in cosymplectic manifolds. These results are part of a manuscript in collaboration with S.L. Druta-Romaniuc, J. Inoguchi, M.I. Munteanu, dealing with the study of magnetic curves in Sasakian and cosymplectic manifolds, in particular also in product spaces $M^2 \times \mathbb{R}$. We proved that a magnetic curve in a Sasakian manifold (and then in a cosymplectic manifold) of arbitrary dimension is a helix of order 3 having the curvatures $k_1 = q \mid \sin \theta \mid$ and $k_2 = q \mid \cos \theta - 1 \mid$, where $\theta$ represents the angle between the curve (the unit tangent vector in every point of the curve) and the characteristic vector field (called also the Reeb vector field) involved in the Sasakian structure (cosymplectic structure, respectively).

In the present project we continued this study for magnetic curves in quasi-Sasakian manifolds regarded as a product type manifold, between a Sasaki and a Kaehler manifold. At the conference „XVIII Geometrical Seminar“ which took place in Vrnjačka Banja, Serbia, 25-28.05.2014, the project leader gave an oral presentation containing some of the results obtained in the study of magnetic curves in quasi-Sasakian manifolds of product type. A manifold defined as the product between a $(2p+1)$-dimensional Sasaki manifold and a $(2k)$-dimensional Kaehler manifold is endowed with a natural quasi-Sasakian structure.

First, we proved that the magnetic curves in $\mathbb{R}^5$, regarded as the product between the Heisenberg group $\mathbb{R}^3(-3)$ endowed with a Sasakian structure and the plane $\mathbb{R}^2$ thought as a Kaehler manifold, have maximum order 5. These results were included in the article [5], published in December 2013.

Second, we generalized these results for arbitrary dimension, classifying magnetic curves in $\mathbb{R}^{2(n+p)+1}$, where the magnetic field is proportional to the fundamental 2-form. We proved that these magnetic curves are helices of order 5 situated in a space $\mathbb{R}^5$, which has the quasi-Sasakian structure induced from the ambient manifold $\mathbb{R}^{2(n+p)+1}$. These results, included in [2], were also presented as a poster at the ICM- International Congress of Mathematicians, Seoul, South Korea, 2014.

Finally, in the article [3] we show that a magnetic curve on the Sasakian sphere $S^{2n+1}$ is found on a totally geodesic sphere $S^3$ whose Sasaki structure is the one induced from $S^{2n+1}$.

**Dissemination: conferences, seminars, foreign researchers**

The results were discussed and presented during the research visits of the project leader in the seminars organized by the host institutions, as follows:
- Department of Mathematics, Instituto de Matematica, Universidade Federal da Bahia, Salvador, Brazil, on 6.08.2013, title of the talk: “Contributions to the study of constant angle surfaces”.
- Departamento de Matematicas, Facultad de Ciencias, Universidad Nacional Autonoma de Mexico, Mexico City, Mexic, on 6.11.2013, title of the talk: “Classification results in the study of constant angle surfaces”.

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and also during the following conferences:

- 29-th Brazilian Colloquium of Mathematics, IMPA, Rio de Janeiro, Brazil, July 2013,
  Title of the poster: “Complete classification of magnetic trajectories in cosymplectic manifolds”.

- The 13th International Conference of Tensor Society on Differential Geometry and its Applications, and Informatics besides, Iasi, September 2013,
  Title of the talk: “On the geometry of constant angle surfaces”.

- XVIII Geometrical Seminar, Vrnjačka Banja, Serbia, 25-28.05.2014;
  Title of the talk: „Magnetic curves in quasi-Sasakian manifolds“.

- ICM- International Congress of Mathematicians, Seoul, South Korea, 13-21.08.2014;
  Title of the poster: „Magnetic curves in quasi-Sasakian manifolds”.

Foreign researchers:

- Oscar Palmas, Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México (UNAM), Mexico City, Mexic, 14.05-03.06.2014.
  Title of the talk: „Foliations of Lorentzian manifolds by $k$-umbilical hypersurfaces”.

- Luc Vrancken, Laboratoire de Mathématiques et ses Applications de Valenciennes (LAMAV), Université de Valenciennes et du Hainaut Cambrésis, Valenciennes, France, 3-10.06.2014.
  Title of the talk: „Flat almost complex surfaces in the nearly Kähler $S^3\times S^3$”.

Miscellanea: In the period 30.06.2013-03.07.2013 the project leader attended the mini-course „Ricci Flow lessons”, given by Prof. Ovidiu Munteanu, University of Connecticut-Storrs, USA, organized by the Transilvania University from Brasov, Romania. The project leader participated also at the following conferences:

- Congreso Nacional de la Sociedad Matematica Mexicana, Universidad Autonoma de Yucatan (UADY), Merida, Mexic, in the period 27.10.2013-03.11.2013.

- ICM Satellite Conference on Real and Complex Submanifolds”, NIMS, Daejeon, South Korea, 10-12.08.2014.


As a member of the Department of Mathematics and Informatics, the project leader has also teaching duties, obtaining in March 2014 a teaching assistant position starting with October 1st, 2014. She is giving a course in mathematics and exercise classes for students in the first year at the Faculty of Architecture and Faculty of Civil Engineering from the „Gh. Asachi” Technical University of Iasi.

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